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Anisotropic surface waves under a vertical magnetic force

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A conducting liquid with a free surface is subjected to a vertical magnetic force due to imposed, horizontal, magnetic and current fields in the liquid. Because the current field is modified differently by differently oriented surface waves, the propagation of gravity waves becomes strongly anisotropic. The cases of shallow and deep fluid are explored. The group velocity shows features that are reminiscent of magnetoacoustic waves. The need for stability of the surface sets limits on the magnetic force which may be imposed. The feasibility of experiments is discussed and the effect of ohmic damping and surface tension is found to be relatively unimportant under suitably chosen conditions.

1. Introduction

Addition of electrical conductivity and magnetic fields to a classical problem in fluid dynamics sometimes, though not always, produces interesting and unexpected effects. A case in point is provided by the gravity waves discussed in this paper. The waves are propagating on the free surface of a conducting liquid in the presence of a vertical body force due partly to gravity and partly to imposed, horizontal, magnetic and current fields in the fluid. The resulting magnetic force may augment or oppose gravity. The interest of the problem arises from the extreme anisotropy of the propagation of waves when the magnetic $\mathbf{j} \times \mathbf{B}$ force is strong enough. As well as being sometimes dispersive, in the normal manner, the waves travel at different phase velocities in different directions, and so the group velocity is also strongly anisotropic. The behaviour is somewhat reminiscent of that predicted theoretically for acoustic waves in perfectly conducting gases in magnetic fields. However, these surface waves should be easily demonstrable in the laboratory. The paper discusses conditions for possible experiments in which the theoretical approximations would be reasonably valid.

The basic reason for the anisotropic propagation is that the waves tilt the current and hence the magnetic force to a variable extent that depends on the direction of propagation. If one thinks of the mechanics of ordinary gravity waves as being essentially the acceleration tangentially of the liquid on the flanks of waves by the tangential component of gravity (the pressure gradient along the surface being zero in the absence of surface tension), then it is clear that a magnetic force that stays vertical as the wave passes will act to help or hinder gravity, and thus alter the wave speed, whereas a magnetic force that stays normal to the surface will have no effect. Now the imposed current component parallel to the wave crests (if these are straight) stays straight and horizontal and so its associated $\mathbf{j} \times \mathbf{B}$ force stays vertical, if we assume the horizontal magnetic field $\mathbf{B}$ is unperturbed by the motion. On the other hand, the current component parallel to the wave normal has to follow the surface, and its associated $\mathbf{j} \times \mathbf{B}$ force stays normal to the surface and has no effect on the waves. The implication is that only the component of imposed current parallel to the wave crests and the associated perpendicular component of magnetic field affect the propagation. Note that the total imposed field and current need not be perpendicular and may even be parallel, whereupon $\mathbf{j} \times \mathbf{B}$ vanishes in the quiescent state. These qualitative conclusions are confirmed by the analysis which follows.

At first sight one might guess that the magnetic force affected the waves principally *via* the current component parallel to the wave normal, which would be intensified under the troughs and weakened under the crests, but this is an entirely misleading approach, because the pressure easily balances the resulting $\mathbf{j} \times \mathbf{B}$ forces in this case.

The case with the imposed $\mathbf{j}$ and $\mathbf{B}$ fields parallel has been briefly discussed by Murty (1961) in his paper on the stability of the free surface or surfaces of a layer of fluid bearing a uniform imposed current. Most of that paper is devoted to discussing the case where the magnetic field is solely due to the imposed current. Northrup (1907) performed experiments in NaK related to this case and appears to have been the first to exploit a covering of a lighter, non-conducting liquid (kerosine) to make it easier for magnetic forces to compete with gravity.

In the present work, however, we take the case where the field due to the imposed current is negligible.

No mention has yet been made of wave damping by ohmic dissipation, which has been discussed previously by Fraenkel (1960) for the case where no current was imposed and the magnetic field was vertical. It will be assumed herein that perturbation of the imposed current by the induced e.m.fs is negligible and this eliminates the ohmic damping. The current is perturbed purely geometrically by the wavy surface. We also neglect all perturbation of the imposed magnetic field. We are thus concerned with a very degenerate, albeit physically realistic, variety of magnetohydrodynamics. Section 5 gives a discussion of feasible physical magnitudes, and the effect of ohmic damping is considered further in the appendix.

One constraint on the severity of the anisotropic effect is that the liquid must be in stable equilbrium in the absence of waves. The paper therefore includes the necessary discussion of this instability and the resulting limits on imposed current and field. Baker (1965) has reported experiments on the instability of the surface in the case where the imposed field and current are mutually perpendicular.

2. Linearized theory (plane waves)

We shall consider small amplitude waves for which the equations are linearizable in the perturbations. The undisturbed state consists of uniform liquid at rest under the horizontal magnetic and current fields. Any distribution of waves can be Fourier analyzed into superposable plane waves. The analysis may thus deal initially with plane, harmonic waves of horizontal wave-numbers k and frequency ω.

Denote vertical distance from the equilibrium surface level by z, and horizontal distance parallel and perpendicular to the wave normal by n and s respectively (see figure 1). Let n, s, z be right-handed co-ordinates. The problem is now two-dimensional, with no variation in the s direction.

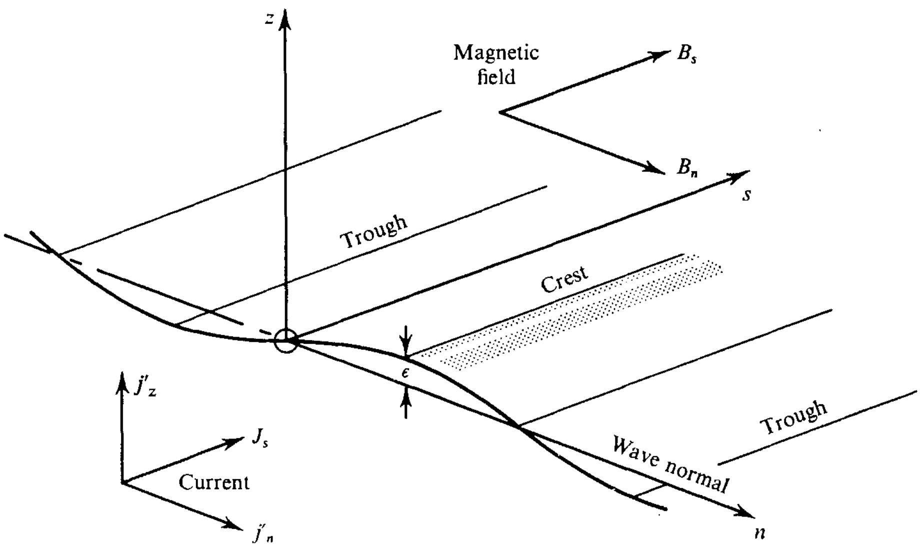

FIGURE 1

In the undisturbed state, we impose a uniform, horizontal magnetic field $\mathbf{B}$ with components B_n and B_s, and a uniform, horizontal current of density $\mathbf{J}$, with components J_n and J_s. In the absence of significant magnetic field perturbations, curl $\mathbf{E} = 0$ ($\mathbf{E}$ being electric field) and so E_s is independent of n and z. If only d.c. voltages are applied to the fluid to drive the imposed current, E_s must be independent of time t also. Neglect of $\mathbf{v} \times \mathbf{B}$ terms in Ohm's law (an approximation which is defended later) implies that the current density in the s direction stays constant at the value J_s.

Taking div $\mathbf{j} = 0$ and assuming $\mathbf{B}$ stays uniform, we have

$$\operatorname{curl} \mathbf{j} \times \mathbf{B} = (\mathbf{B}.\operatorname{grad})\mathbf{j}, \tag{1}$$

which has no s component. The linearized equation of fluid motion is

$$\rho\, \partial \mathbf{v}/\partial t + \operatorname{grad} p = \mathbf{j} \times \mathbf{B} - \rho g \mathbf{i} \quad (\mathbf{i} \text{ a vertical unit vector}) \tag{2}$$

if viscosity is neglected. Thus $\rho\, \partial \boldsymbol{\omega}/\partial t = \operatorname{curl} \mathbf{j} \times \mathbf{B}$, if $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$, the vorticity, and in particular $\partial \omega_s/\partial t = 0$. Since all perturbation quantities, including vorticity, are harmonic functions of time, ω_s must vanish, and we can take the flow in the z, n plane as being irrotational, with two-dimensional velocity potential $\phi(z, n, t)$. Incompressibility then implies

$$\nabla^2 \phi = 0, \tag{3}$$

the Laplacian being two-dimensional.

The current flow $\mathbf{j}'$ in the z, n plane is perturbed by the waviness of the surface, in general. The vertical perturbation current component j'_z produces an oscillating magnetic force $j'_z B_n$ which produces motions in the s direction, but these have little importance in the present approximation. It is unnecessary to calculate the current distribution.

In equation (2), $\mathbf{j} \times \mathbf{B}$ can be expressed as $-J_s B_n \mathbf{i}$ plus $\mathbf{j}' \times \mathbf{B}_s$ (both of which lie in the z, n plane) plus the s component. Note that at the liquid surface, $\mathbf{j}'$ must be parallel to the surface and hence $\mathbf{j}' \times \mathbf{B}_s$ is normal to it. Equation (2) can be partially rewritten

$$\mathrm{grad}\,\{\rho\,\partial\phi/\partial t + p + (\rho g + J_s B_n) z\} = \mathbf{j}' \times \mathbf{B}_s,$$

where grad is two-dimensional. Integrating along the surface, $z_0 = \epsilon\, e^{i(\omega t - kn)}$, say, we deduce that

$$\rho\,\partial\phi/\partial t + p + (\rho g + J_s B_n) z_0 = F(t) \quad \text{there.} \tag{4}$$

If the ambient pressure above the liquid is taken as zero, and surface tension is α, then in the liquid at the surface, $p = -\alpha\,\partial^2 z_0/\partial n^2 = k^2 \alpha \epsilon\, e^{i(\omega t - kn)}$, if we assume that $\partial z_0/\partial n$ is small, i.e. ϵ is small in comparison with the wavelength. We apply the boundary conditions at $z = 0$ in the usual manner. Since ϕ is also periodic in n, evidently $F(t) = 0$ in (4), which becomes

$$\rho\,\partial\phi/\partial t + (k^2\alpha + \rho g + J_s B_n)\,\epsilon\, e^{i(\omega t - kn)} = 0 \quad \text{at} \quad z = 0. \tag{5}$$

The other surface boundary condition is that

$$v_z = \partial\phi/\partial z = Dz_0/Dt = \partial z_0/\partial t \tag{6}$$

to the first order, at $z = 0$.

If the fluid is of finite uniform depth h, there is the further condition $\partial\phi/\partial z = 0$ at $z = -h$. To satisfy this condition, (6) and (3), ϕ must be of the form

$$\phi = (i\omega\epsilon/k)\, e^{i(\omega t - kn)}\{\sinh kz + \coth kh \cosh kz\}.$$

This also satisfies (5) if

$$\rho\omega^2 = k \tanh kh (k^2\alpha + \rho g + J_s B_n), \tag{7}$$

the dispersion relation. This is consistent with Murty's (1961) earlier dispersion relation in the case which is common to the present work, namely: $\beta = 0$, one free surface and negligible magnetic field due to the imposed current. It should be noted that magnetohydrodynamic effects only enter our problem *via* the boundary condition on pressure.

The anisotropy resides in the term $J_s B_n$ in (7). As remarked earlier, only J_s and B_n affect the waves. If the angle between $\mathbf{J}$ and $\mathbf{B}$ is β, and the wave normal and wave-number vector $\mathbf{k}$ are both inclined at θ to $\mathbf{B}$ and the x axis, as in figure 2, then

$$J_s B_n = JB \cos\theta \sin(\beta - \theta),$$

the θ dependent term in (7). The waves are also dispersive with respect to frequency unless the phase velocity $c = \omega/k$ is independent of ω or k. This only occurs when surface tension is negligible and the liquid is shallow compared with the wavelength, with $kh \ll 1$ and $\tanh kh \doteqdot kh$.

Before we study the significance of (7), it is necessary to formulate the limits on the anisotropic $J_s B_n$ term that arise from the need for the undisturbed state to be stable. In the common case $\beta = 0$, our results agree with Murty's stability condition, with self-field neglected.

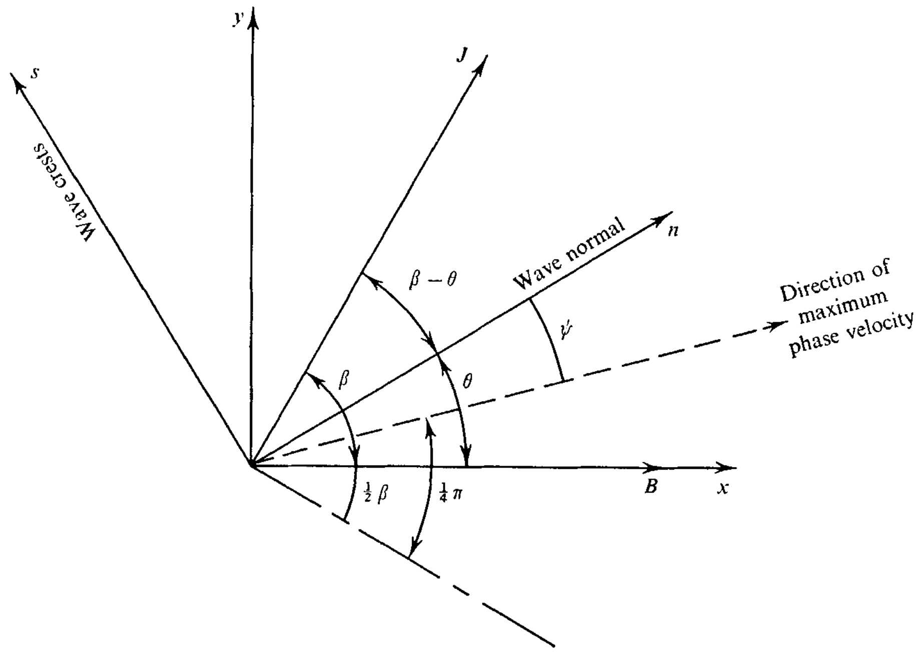

Figure 2. Top view of liquid surface.

3. Stability considerations

The preceding analysis can be utilized for a study of stability, subject to the same approximations. Instability corresponds to the case where (7) permits $i\omega$ to have a real positive part for at least one real value of k and θ. Since then ω^2 is real, $i\omega$ would have to be wholly real, i.e. ω^2 negative, with $k^2\alpha + \rho g + J_s B_n < 0$, for some value of θ. Thus stability requires $k^2\alpha + \rho g + J_s B_n > 0$ for all k and θ, or

$$\rho g + J_s B_n > 0, \quad \text{for all } \theta, \tag{8}$$

to achieve stability of even long waves (of small k) which surface tension cannot stabilize.

From this point onwards, we shall assume that $k^2\alpha \ll \rho g$, so that surface tension may be neglected, to bring out more clearly the magnetic effects. The dispersion relation is then

$$\rho\omega^2 = k \tanh kh\{\rho g + JB\cos\theta \sin(\beta - \theta)\}. \tag{9}$$

Let ω_0 be the frequency of waves of the same wave-number in the absence of magnetic effects, where

$$\omega_0^2 = gk \tanh kh. \tag{10}$$

Then

$$\omega^2/\omega_0^2 = 1 + \gamma \cos\theta \sin(\beta - \theta), \tag{11}$$

if $\gamma = JB/\rho g$; an essentially positive ratio. Stability requires that the right-hand side of (11) be positive for all values of θ, i.e. that $\gamma < 2/(1-\sin\beta)$. $\mathbf{J}\times\mathbf{B}$ helps gravity if $0 < \beta < \pi$ and opposes it if $\pi < \beta < 2\pi$, if we let β range from 0 to 2π. Equation (11) is usefully rewritten

$$\omega^2/\omega_0^2 = (1+\tfrac{1}{2}\gamma\sin\beta)(1+\delta\cos 2\psi), \tag{12}$$

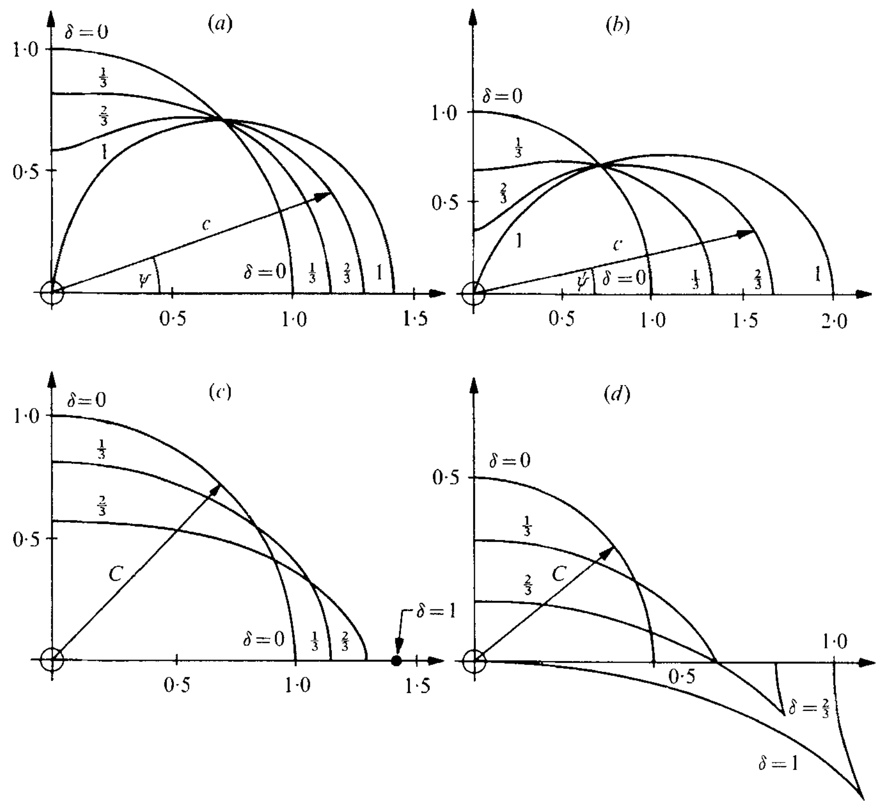

FIGURE 3. Polar plots of phase velocity (a) and (b) and group velocity (c) and (d) for shallow liquid (a) and (c) and deep liquid (b) and (d).

where $\delta = \tfrac{1}{2}\gamma/(1+\tfrac{1}{2}\gamma\sin\beta)$ and $\psi = \theta - \tfrac{1}{2}\beta + \tfrac{1}{4}\pi$. The point of these substitutions is that we have reduced the purely θ or ψ dependent factor to a form that depends on only one parameter δ for various values of γ and β. Stability now requires that

$$1 > \delta > 0. \tag{13}$$

When $\beta = \tfrac{1}{2}\pi$, and $\mathbf{J}\times\mathbf{B}$ helps gravity, there is seen to be no limit on the magnitude of JB, but in all other cases JB is restricted, even when $\mathbf{J}\times\mathbf{B}$ is zero. The stability limit is a stronger constraint than the requirement that gravity can overcome $\mathbf{J}\times\mathbf{B}$, so that the fluid is not lifted bodily when $\sin\beta$ is negative. This more primitive requirement is merely $1+\gamma\sin\beta > 0$ or $\gamma < 1/(-\sin\beta)$, i.e. weaker than the stability condition. The two conditions only coincide for $\beta = \tfrac{3}{2}\pi$, with J and B perpendicular and JB limited to ρg. In Murty's case with J and B parallel or anti-parallel, and no magnetic force in the undisturbed state, the limit on JB is $2\rho g$, whether β is 0 or π.

4. Phase velocity (surface tension neglected)

The phase velocity c or ω/k in the direction θ in general depends on ω or k as well as θ (or ψ). The exception is the case of shallow fluid, with $kh \ll 1$ and $\omega_0 = k^2gh$, so that

$$c^2 = gh(1+\tfrac{1}{2}\gamma\sin\beta)(1+\delta\cos 2\psi), \tag{14}$$

irrespective of ω or k.

For other cases, it is most fruitful to think of ω as fixed and then consider c as a function of ψ. For deep fluid, with $kh \gg 1$, $\omega_0^2 \doteqdot gk$ and

$$c = (g/\omega)(1+\tfrac{1}{2}\gamma\sin\beta)(1+\delta\cos 2\psi). \tag{15}$$

Figure 3 shows one quadrant of the phase velocity plotted as a radius vector as a function of its inclination ψ to the direction of maximum phase velocity, namely, $\theta = \frac{1}{4}\pi - \frac{1}{2}\beta \,(\mathrm{mod}\,\pi)$, which is related quite subtly to the directions of J and B, as figure 2 shows. For the most 'natural' cases, with $\beta = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, $\theta = 0$ or $\frac{1}{2}\pi\,(\mathrm{mod}\,\pi)$ respectively. The scales in figure 3 are such that phase velocity is given as a multiple of $\{gh(1+\frac{1}{2}\gamma\sin\beta)\}^{\frac{1}{2}}$ in figure 3(*a*) (shallow liquid) and of $(g/\omega)(1+\frac{1}{2}\gamma\sin\beta)$ in figure 3 (*b*) (deep liquid). The curves are plotted for values of δ equal to 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1 (the stability limit, at which $c_{\min}$ becomes zero). It is worth noting that for $\delta = 1$, in the case of shallow fluid, the c plot in each quadrant is a semicircle, exactly as for Alfvén waves. The complete c plot in each case can of course be found by reflexion of figure 3 in its axes.

5. The validity of the approximations and the experimental prospects

Linearization involved the neglect of the quadratic $(\mathbf{v}.\mathrm{grad})\mathbf{v}$ term in comparison with the $\partial\mathbf{v}/\partial t$ and other terms. As with ordinary waves, this is valid provided $k\epsilon \ll 1$, i.e. amplitude $\ll$ wavelength, and $\epsilon \ll h$, i.e. amplitude $\ll$ depth.

We have also assumed the magnetic field uniform, unperturbed by both the imposed current and the induced currents due to $\mathbf{v}\times\mathbf{B}$. A reasonable assumption is that the liquid occupies a trough of length and breadth l to depth h, the imposed current J being supplied between a pair of highly conducting opposing faces. The other faces of the trough are non-conducting. Probably h would be less than l, and much smaller than l if shallow fluid waves were under investigation. It would be reasonable to return the total current Jlh *via* a flat conductor immediately below the trough to minimize stray field due to J. Ignoring edge effects, the field due to J in the fluid would then be horizontal and perpendicular to J, rising from zero at the surface to a value μJh at the bottom. The irrotationality of the motion in the z, n planes is destroyed by the presence of such a non-uniform field. In a finite experiment, moreover, edge effects would probably occur and so the perturbation field of order μJh due to J should be kept small, i.e. $\mu Jh \ll B$.

In discussing the further approximations, we take first the case of deep liquid, pausing finally to note the differences that occur with shallow liquid.

Deep liquid ($kh \gg 1$). Consider the effects of the e.m.f. $\mathbf{v}\times\mathbf{B}$, which has been neglected in the analysis. Associated with it will be an induced current contribution which perturbs the magnetic field, with consequent non-zero curl **E**. The induced current also produces a magnetic damping force. Only the rotational part of $\mathbf{v}\times\mathbf{B}$ can drive currents in the fluid, namely that part due to B_n. So the induced currents will be of order $\sigma v_z B_n$ in the s direction and $\sigma v_s B_n$ in the z, n planes; while v_z and v_s are both of order $\omega\epsilon$. (v_s is of order $k\epsilon JB/\rho\omega$, which is of order $\omega\epsilon$, if we take JB of order ρg, for interesting effects, and ω^2 of order kg.) The currents of order $\sigma\omega\epsilon B$ cause perturbation fields of order $\mu\sigma\omega\epsilon B/k$ (since $1/k$ is the relevant length scale) which are negligible if $\mu\sigma\omega\epsilon/k \ll 1$. This is also the condition for neglecting the modification of the induced currents due to the fact that curl $\mathbf{E} \neq 0$. Forces associated with the induced currents could affect the dynamics by disrupting the two-dimensional potential flow or the boundary condition. In both cases it is only s-wise current and the associated magnetic field in z, n planes that matter. For small effects on the boundary conditions we need $\sigma\omega\epsilon B \ll J$, and for small effects on the bulk flow we need the induced $\mathbf{j}\times\mathbf{B}$ forces, of order $\sigma\omega\epsilon B^2$, to be small compared with $\rho\,\partial\mathbf{v}/\partial t$, of order $\rho\epsilon\omega^2$, i.e. $\sigma B^2 \ll \rho\omega$. This includes and surpasses the previous condition since $k\epsilon \ll 1$, $\omega^2 \approx kg$ and $JB \approx \rho g$.

Another effect is that the wavy surface involves current perturbations in z, n planes, and hence B_s perturbations that produce a non-zero curl **E** in the s direction and further modify the current in z, n planes. The currents due to waviness are of order $k\epsilon J$, causing magnetic perturbations of order $\mu\epsilon J$, whose changing causes electric fields of order $\omega\mu\epsilon J/k$ and currents of order $\omega\mu\sigma\epsilon J/k$. These are negligible in comparison with $k\epsilon J$ if $\omega\mu\sigma/k^2 \ll 1$. The condition may be interpreted as requiring skin depth $\gg$ wavelength.

Viscosity is negligible if $\rho\nu\nabla^2\mathbf{v}$, of order $\rho\nu k^2 v$, is much smaller than $\rho\,\partial\mathbf{v}/\partial t$, of order $\rho\omega v$, i.e. if $\nu k^2 \ll \omega$, ν being kinematic viscosity. The last two conditions are compatible since, for real liquids, $\mu\sigma\nu \ll\ll 1$. If we also wish to retain the simplification that surface tension is negligible, we require $k^2\alpha \ll \rho g$, assuming $J_s B_n + \rho g$ is of order ρg.

Shallow liquid ($kh \ll 1$). In this case there are differences because ω^2 is of order $k^2 gh$, and the effects of induced currents are weaker because of the shallowness. The perturbation fields due to the induced currents in the s direction are of order $\mu\sigma\omega\epsilon Bh$ and we now require $\mu\sigma\omega\epsilon h \ll 1$. We retain the condition $\sigma\omega\epsilon B \ll J$ now. The condition for small disruption of the potential flow by induced $\mathbf{j}\times\mathbf{B}$ forces is now weaker than $\sigma B^2 \ll \rho\omega$, because the forces, vertical and of order $\sigma\omega\epsilon B^2$, must not be compared merely with $\rho\,\partial v_z/\partial t$, since this is negligible in shallow liquid. The curl of the force, of order $k\sigma\omega\epsilon B^2$, produces vorticity of order $k\sigma\epsilon B^2/\rho$, consisting chiefly of $\partial v_n/\partial z$, so that the v_n perturbations are of order $k\sigma\epsilon B^2 h/\rho$. These are small compared with v_n in the irrotational solution (of order $\omega\epsilon/kh$) if $\sigma B^2 k^2 h^2/\rho\omega \ll 1$. This condition for the neglect of damping is confirmed more rigorously in the appendix.

The condition for neglecting currents induced by curl **E** from the changing magnetic field associated with waviness of the current pattern now becomes $\omega\mu\sigma h/k \ll 1$ because of the restricted depth.

Possible experimental conditions

That the theoretical approximations we made are reasonable can be seen by considering some typical, feasible experimental conditions. The most convenient fluid is mercury; selecting a better conductor like sodium would be ill-advised because the damping would be higher. With mercury, approximate values are:

$$\sigma = 10^6 \text{ per } \Omega\text{m}, \qquad \rho = 1{\cdot}4 \times 10^6 \,\text{kg/m}^3,$$

$$\nu = 10^{-7}\,\text{m}^2/\text{s}, \qquad \alpha = 0{\cdot}5\,\text{N/m}.$$

Also $$\mu = 1{\cdot}2 \times 10^{-6}\,\text{Wb/A-m}, \quad g = 10\,\text{m/s}^2.$$

Let $B = 0{\cdot}2\,\text{Wb/m}^2$ and $J = 5 \times 10^5\,\text{A/m}^2$, both feasible practically, although a large low-voltage current source would be needed. Then $\gamma = JB/\rho g = 0{\cdot}7$, so that interesting effects should occur. B must not be too big to keep the damping low. One limit on J is ohmic heating of the mercury. If we ignore heat losses, its temperature would rise at a rate of 0·12 °C per second with the above value of J. This would be acceptable if runs were not protracted. However, the values of J which Murty (1961) suggests for experiments with self-field only are up to $10^7\,\text{A/m}^2$, which would imply unacceptably high rates of heating.

Deep liquid. Consider first an experiment in 'deep' liquid, of depth 0·05 m. For present purposes take $kg = \omega^2$, and consider waves excited at a frequency $\omega = 30\,\text{s}^{-1}$. Then $k = 90\,\text{m}^{-1}$. The crucial ratios, which should be reasonably small for our approximations to be valid, are then as follows:

deep fluid approximation: $1/kh = 0{\cdot}22$;

(for this value $\tanh kh$ is within 0·02 % of unity);

small effect of J on B: $\mu J h/B = 0{\cdot}015$;

small damping by induced j: $\sigma B^2/\rho\omega = 0{\cdot}09$;

negligible surface tension: $\alpha k^2/\rho g = 0{\cdot}03$;

small curl $\mathbf{E}$: $\omega\mu\sigma/k^2 = 0{\cdot}004$;

negligible viscosity: $\nu k^2/\omega = 3 \times 10^{-5}$.

It is the first four ratios in the above list which really constrain the choice of conditions.

In addition there are constraints on the amplitude ϵ:

linearization condition: $\epsilon \ll 1/k = 0{\cdot}011\,\text{m}$;

small effect of motion on B: $\epsilon \ll k/\mu\sigma\omega = 2{\cdot}5\,\text{m}$.

The first condition dominates, but could easily be met.

Shallow liquid. Somewhat different and less stringent conditions must be met for an experiment in shallow mercury. Now we select $h = 0{\cdot}005\,\text{m}$ and $k = 40\,\text{m}^{-1}$. If we take $\omega^2 = k^2gh$, for present purposes, then $\omega = 9\,\text{s}^{-1}$. The crucial ratios, which should be reasonably small for our approximations to be valid, are then as follows:

shallow fluid approximation:	$kh = 0{\cdot}2$
(for this value, $\tanh kh = kh$ to within 1·3 %);	
small effect of J on B:	$\mu J h/B = 0{\cdot}015$;
small damping by induced j:	$\sigma B^2 k^2 h^2/\rho\omega = 0{\cdot}013$;
negligible surface tension:	$\alpha k^2/\rho g = 0{\cdot}0057$;
small curl **E**:	$\omega\mu\sigma h/k = 0{\cdot}0013$;
negligible viscosity:	$\nu k^2/\omega = 2\times 10^{-5}$.

In addition there are constraints on the amplitude ϵ:

linearization condition:	$\epsilon \ll h = 0{\cdot}005$ m;
small effect of motion on B:	$\epsilon \ll 1/\mu\sigma\omega h = 200$ m;
small effect on dynamic boundary condition	$\epsilon \ll J/\sigma\omega B = 0{\cdot}25$ m.

The first condition dominates, but could easily be met.

With stronger fields it would also be quite feasible to run experiments with electrolytes, and the damping of the waves would then be extremely low.

Choice of β

The parameter δ should be made as large as possible within the stability limit for most interesting effects. The question then arises as to what value of β is most advantageous in order to avoid excessive values of JB. It emerges that the most economical values of JB and γ are achieved by taking $\beta = \frac{3}{2}\pi$. Then $\gamma = 1$ makes $\delta = 1$. Thus the most obvious configuration, that with J and B perpendicular, is also the best. The direction of maximum phase velocity is then $\theta = \pm\frac{1}{2}\pi$, i.e. in the $\pm J$ direction.

6. General propagation and group velocity

The anisotropy of the phase velocity displayed in figure 3 implies that energy will propagate with an anisotropic group velocity, which is not in general parallel to the phase velocity. The dispersion relation $\omega = f(k,\psi)$ can be replaced by $\omega = F(k_1,k_2)$ where $k_1 = k\cos\psi$ and $k_2 = k\sin\psi$, k_1 and k_2 being the components of the wave-number vector. Then the components of the group velocity **C** are $C_1 = \partial\omega/\partial k_1$, $C_2 = \partial\omega/\partial k_2$ (Whitham 1961).

Shallow liquid. Here we have

$$\omega^2 = k^2 gh(1+\tfrac{1}{2}\gamma\sin\beta)(1+\delta\cos 2\psi)$$

or

$$\omega^2 = gh(1+\tfrac{1}{2}\gamma\sin\beta)\{(1+\delta)k_1^2+(1-\delta)k_2^2\}.$$

It follows that

$$\frac{C_1}{\{gh(1+\frac{1}{2}\gamma\sin\beta)\}^{\frac{1}{2}}} = \frac{(1+\delta)\cos\psi}{(1+\delta\cos 2\psi)^{\frac{1}{2}}} \quad\text{and}\quad \frac{C_2}{\{gh(1+\frac{1}{2}\gamma\sin\beta)\}^{\frac{1}{2}}} = \frac{(1-\delta)\sin\psi}{(1+\delta\cos 2\psi)^{\frac{1}{2}}}.$$

These results are presented as a polar plot in figure 3 (c). The scale is such that C is expressed as a multiple of $\{gh(1+\frac{1}{2}\gamma\sin\beta)\}^{\frac{1}{2}}$. Note that ψ is the inclination of the related wave-number vector, not of **C** itself. The curves in figure 3(c) are, somewhat surprisingly, ellipses, because

$$C_1^2/(1+\delta)+C_2^2/(1-\delta) = gh(1+\tfrac{1}{2}\gamma\sin\beta).$$

When $\delta = 1$ (the stability limit), $C_1 = \pm\{2gh(1+\frac{1}{2}\gamma\sin\beta)\}^{\frac{1}{2}}$ and $C_2 = 0$, and the locus degenerates to two points on the $\psi = 0$ axis. There is then unidirectional propagation as in Alfvén waves, but at an angle $(\frac{1}{4}\pi - \frac{1}{2}\beta)$ to the magnetic field.

Deep liquid. In this case

$$\omega^2 = gk(1+\tfrac{1}{2}\gamma\sin\beta)(1+\delta\cos 2\psi),$$

or

$$\omega^2 = g(1+\tfrac{1}{2}\gamma\sin\beta)\{(1+\delta)k_1^2+(1-\delta)k_2^2\}/(k_1^2+k_2^2)^{\frac{1}{2}}.$$

It follows that

$$\frac{\omega C_1}{g(1+\frac{1}{2}\gamma\sin\beta)} = \tfrac{1}{2}(1+2\delta-\delta\cos 2\psi)\cos\psi,$$

and

$$\frac{\omega C_2}{g(1+\frac{1}{2}\gamma\sin\beta)} = \tfrac{1}{2}(1-2\delta-\delta\cos 2\psi)\sin\psi.$$

These results are presented as a polar plot in figure 3(*d*). The scale is such that C is expressed as a multiple of $(g/\omega)(1+\frac{1}{2}\gamma\sin\beta)$. The character of the locus changes markedly when δ exceeds the value $\frac{1}{3}$. Then C_1 and C_2 experience a simultaneous extremum as ψ varies and cusps result. There is some resemblance to the behaviour of slow magneto-acoustic waves, but there are no cusps on the $\psi = 0$ axis at non-zero C as in the magneto-acoustic case.

Figures 3 (*c*) and (*d*) only present one quadrant of the plot, which may be completed by reflexion in the axes.

The results are novel enough to suggest that experiments in both deep and shallow liquid would be rewarding, despite the obvious experimental difficulties, and the invalidity of our approximations under some circumstances. Investigations could include plane waves, standing wave modes, resonance, and waves excited at a point harmonically or by an impulse. The shallow liquid case is simpler in that group and phase velocities are independent of frequency, whereas the case of deeper liquid provides a good manifestation of the class of linear waves which are dispersive with respect both to frequency and orientation.

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Appendix. The importance of the $\mathbf{v}\times\mathbf{B}$ e.m.f. and damping

In the main paper the effect of $\mathbf{v}\times\mathbf{B}$ on the current flow was ignored. To exhibit its degree of importance, we now investigate its effect in the simple particular case where $\mathbf{J}$ and $\mathbf{B}$ are perpendicular and one of them is parallel to the (straight) wave crests.

If $\mathbf{B}$ is parallel to the wave crests, curl $\mathbf{v}\times\mathbf{B} = 0$, if we assume $\mathbf{B}$ uniform and unperturbed, and $\mathbf{v}\times\mathbf{B}$ cannot drive currents and is balanced by an electric field. In this configuration there is no damping of waves by the magnetic force.

We therefore turn to the case where $\mathbf{J}$ is parallel to the wave crests (the s direction) and $\mathbf{B}$ lies in the n direction. If we do not neglect $\mathbf{v}\times\mathbf{B}$, the current density j_s is not uniform and equal to J_s, as was assumed in §2. Equation (1) then indicates that the s component of $\operatorname{curl}\mathbf{j}\times\mathbf{B}$ is $B(\partial j_s/\partial n)$ which equals $\sigma B^2\partial v_z/\partial n$ when we keep the $\mathbf{v}\times\mathbf{B}$ term in Ohm's Law,

$$\mathbf{j}/\sigma = \mathbf{E}+\mathbf{v}\times\mathbf{B},$$

E_s being uniform. Since the flow in z, n planes is evidently no longer irrotational, we make use of the two-dimensional stream function ψ instead of ϕ. Then the inviscid, linearized vorticity equation

$$\rho\,\partial\boldsymbol{\omega}/\partial t = \operatorname{curl}\mathbf{j}\times\mathbf{B}$$

has the s component

$$\rho\nabla^2\,\partial\psi/\partial t = -\sigma B^2\,\partial^2\psi/\partial n^2, \tag{A 1}$$

since $v_n = \partial\psi/\partial z$, $v_z = -\partial\psi/\partial n$. The boundary condition at the surface is that $p = \text{const.}$, if we neglect surface tension, or $\partial p/\partial n+(\partial p/\partial z)(\partial z_0/\partial n) = 0$. But, from (2), $\partial p/\partial z = -(\rho g+JB)$, to the first order, and

$$\partial p/\partial n = -\rho\,\partial v_n/\partial t = -\rho\,\partial^2\psi/\partial z\,\partial t.$$

Thus

$$\rho\,\partial^3\psi/\partial t^2\,\partial z = (\rho g+JB)\,\partial^2\psi/\partial n^2 \quad \text{at the surface,} \tag{A 2}$$

since $v_z = -\partial\psi/\partial n = \partial z_0/\partial t$, to the first order. A solution of the form $\psi = e^{i\omega t+mn}(C\cosh kz+D\sinh kz)$ satisfies (A 1) if $\rho i\omega(m^2+k^2) = -\sigma B^2 m^2$, or

$$k \approx \pm im(1-\tfrac{1}{2}i\lambda), \tag{A 3}$$

if the damping factor $\lambda = \sigma B^2/\rho\omega$, is small. It satisfies (A 2) at $z = 0$ if $\rho\omega^2 kD = -(\rho g+JB)m^2 C$. If the depth of liquid is h, the other boundary condition is that $\partial\psi/\partial n = 0$ at $z = -h$. Hence $C = D\tanh kh$ and

$$\rho\omega^2 k = -(\rho g+JB)m^2\tanh kh. \tag{A 4}$$

We shall take ω as real, k and m as complex, and consider just the cases of deep and shallow liquid, for simplicity.

Deep liquid. Here (A 4) becomes $\omega^2 k = -g'm^2$, if $g' = g+JB/\rho$. Using (A 3) we have $m \approx -i(\omega^2/g')(1-\frac{1}{2}i\lambda)$, and it follows that, in a wavelength $2\pi g'/\omega^2$ in the n direction, the wave amplitude falls in the ratio $e^{\pi\lambda}$. With the value $\lambda = 0{\cdot}09$ cited in §5, this ratio is 0·756 and damping is seen to be significant but not disastrous.

Shallow liquid. Here (A 4) becomes $\omega^2 \approx -hg'm^2(1-\frac{1}{3}k^2h^2)$ and

$$m \approx \{i\omega/(g'h)^{\frac{1}{2}}\}(1+k^2h^2/6).$$

Note that a second term in the tanh expansion must be retained to reveal damping. From (A 3) we have $k^2h^2 \approx h\omega^2(1-i\lambda)/g'$, so that

$$m \approx \{i\omega/(g'h)^{\frac{1}{2}}\}(1-i\lambda k^2h^2/6).$$

This confirms that λk^2h^2 or $\sigma B^2k^2h^2/\rho\omega$ is the criterion for small damping. In a wavelength $2\pi(g'h)^{\frac{1}{2}}/\omega$ in the n direction, the wave amplitude falls in the ratio $e^{\pi\lambda k^2h^2/3}$, which equals 0·986 for the values cited in §5. Damping is seen to be much weaker in shallow liquid.

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